Section 5: Sensitivity Analysis and the Simplex Tableau

Question: **What can we get out of the Simples Tableau:**

Answer:

1. OPTIMAL BASIC FEASIBLE SOLUTIONS IF THEY EXIST
2. VALUE OF OBJECTIVE FUNCTION AT O.B.F.S.
3. STATUS OF RESOUCES
4. MARGINAL WORTH (SHADOW PRICES)
5. sensitivity of optimum to changes in the parameters of the model (resources, activities, cost coefficients)
6. DUAL SOUTION AND SIMPLEX MULTIPLIERS (next section)

formal proof.

We will use the Reddy Mikks Co. for our example throughout.

Recall, the LP in standard form:

$\min z = -3x_E - 2x_I$ (negative of sales)

s.t.

$$x_E + 2x_I + y_1 = 6 \quad (1) \text{ (mat A)}$$

$$2x_E + x_I + y_2 = 8 \quad (2) \text{ (mat B)}$$

$$-x_E + x_I + y_3 = 1 \quad (3) \text{ (mktng dept –demand)}$$

$$x_I + y_4 = 2 \quad (4) \text{ (mktng dept –demand)}$$

$$x_E \geq 0,\ x_I \geq 0,\ y_i \geq 0\ i=1,\ldots,4$$

At obfs:

Resource	Slack	State of Resource
Mat (A) (1)	$y_1=0$	scarce
Mat (B) (2)	$y_2=0$	scarce
Limit on excess interior paint (3)	$y_3=3$	abundant
Limit on demand exterior paint (4)	$y_4=2/3$	abundant

Rule: If at the o.b.f.s. the slack in a constraint corresponding to a resource is 0, then the resource is scarce. Otherwise it is abundant.

A. Unit Worth of a Resource and Maximum Allowable Change

Recall: $z = z_0 + \sum_{j=m+1}^{n} r_j x_j$ if the basic variables are $x_1, \ldots, x_m$.

Increasing x_j for $j=m+1,\ldots,n$ from 0 to 1 changes z by r_j, ($decreases if $r_j<0$) and if x_j is a slack variable, it could be thought of as decreasing the availability of resource (j).

For example, in the Reddy Mikks Co. example, if change the RHS of each of the constraints by d_1, d_2, d_3 and d_4, then the LP problem becomes

$\min z = -3x_E - 2x_I$ (negative of sales)

s.t.

$x_E + 2x_I + y_1 = 6 + d_1$ (1) (mat A)

$2x_E + x_I + y_2 = 8 + d_2$ (2) (mat B)

$-x_E + x_I + y_3 = 1 + d_3$ (3) (mktng dept – demand)

$x_I + y_4 = 2 + d_4$ (4) (mktng dept – demand)

$x_E \geq 0,\ x_I \geq 0,\ y_i \geq 0\ i=1,\ldots,4$

Adjoining the d_i columns to the tableau gives, for the initial tableau

Basic	x_E	x_I	y_1	y_2	y_3	y_4	$-z$	Solution	$\underline{d}^1$	$\underline{d}^2$	$\underline{d}^3$	$\underline{d}^4$
$-z$	–3	–2	0	0	0	0	1	0	0	0	0	0
y_1	1	2	1	0	0	0	0	6	1	0	0	0
y_2	2	1	0	1	0	0	0	8	0	1	0	0
y_3	–1	1	0	0	1	0	0	1	0	0	1	0
y_4	0	1	0	0	0	1	0	2	0	0	0	1

and noting the sections in dark blue are the same, the final tableau becomes

Tableau 2:

Basic	x_E	x_I	y_1	y_2	y_3	y_4	$-z$	Solution	$\underline{d}^1$	$\underline{d}^2$	$\underline{d}^3$	$\underline{d}^4$
$-z$	0	0	$\frac{1}{3}$	$\frac{4}{3}$	0	0	1	$+12\,{}^{2}/_{3}$	$\frac{1}{3}$	$\frac{4}{3}$	0	0
x_I	0	1	$\frac{2}{3}$	$-\frac{1}{3}$	0	0	0	$\frac{4}{3}$	$\frac{2}{3}$	$-\frac{1}{3}$	0	0
x_E	1	0	$-\frac{1}{3}$	$\frac{2}{3}$	0	0	0	$\frac{10}{3}$	$-\frac{1}{3}$	$\frac{2}{3}$	0	0
y_3	0	0	-1	1	1	0	0	3	-1	1	1	0
y_4	0	0	$-\frac{2}{3}$	$\frac{1}{3}$	0	1	0	$\frac{2}{3}$	$-\frac{2}{3}$	$\frac{1}{3}$	0	1

This is the new optimum tableau and has solution

$$z = -12\frac{2}{3} - \frac{1}{3}d_1 - \frac{4}{3}d_2$$

$$x_I = \frac{4}{3} + \frac{2}{3}d_1 - \frac{1}{3}d_2$$

$$x_E = \frac{10}{3} - \frac{1}{3}d_1 + \frac{2}{3}d_2$$

$$y_3 = 3 - d_1 + d_2 + d_3$$

$$y_4 = \frac{2}{3} - \frac{2}{3}d_1 + \frac{1}{3}d_2 + d_4$$

We see that a unit change in operation 2 ($d_1 = \pm 1$) yields a change in z of $\square\frac{1}{3}$ (thousand) dollars. Note that $d_1 \le 1$ in order that $y_4 \ge 0$. For that change x_E and y_3 will still be feasible, while the feasibility of x_I are not affected by a positive value of d_1 since the coefficients of d_1 in their solutions at the final tableau are positive.) Therefore the largest increase in operation 1 is $\Delta b = 1$ and the marginal worth is $\frac{\Delta z}{\Delta b} = \frac{\frac{1}{3}}{1} = \frac{1}{3}$.

We see that a unit change in operation 2 ($d_2 = \pm 1$) yields a change in z of $\square\frac{4}{3}$ (thousand) dollars. Note that $d_2 \le 4$ in order that $x_I \ge 0$. (The feasibility of x_E, y_3 and y_4 are not affected by a positive value of d_2 since the coefficients of d_2 in their solutions at the final tableau are positive .) Therefore the largest increase in operation 2 is $\Delta b = 4$ and the marginal worth is $\frac{\Delta z}{\Delta b} = \frac{\frac{16}{3}}{4} = \frac{4}{3}$.

We see that a unit change in operation 3 ($d_3 = \pm 1$) yields a change in *z* of 0 (thousand) dollars. Note that $d_3 \geq -3$ in order that $y_3 \geq 0$. The feasibility of x_E, x_I and y_4 are unaffected since d_3 does not appear in their solutions. Therefore the largest decrease in operation 3 is $\Delta b = -3$ and the marginal worth is $\frac{\Delta z}{\Delta b} = \frac{0}{-3} = 0$.

We see that a unit change in operation 4 ($d_4 = \pm 1$) yields a change in *z* of 0 (thousand) dollars. Note that $d_4 \geq -\frac{2}{3}$ in order that $d_4 \geq 0$. Therefore the largest decrease in operation 4 is $\Delta b = -\frac{2}{3}$ and the marginal worth is $\frac{\Delta z}{\Delta b} = \frac{0}{-\frac{2}{3}} = 0$.

Note that the marginal worth of the four resources (dual prices) is the same as the corresponding reduced costs in the final tableau.

Note also that our method of finding changes in resources also allows simultaneous changes which we can confirm as either possible or not.

How did we find the change in resource availability for say, resource 1?

Adding Δ_1 to the RHS of (1) in the original tableau and follow the calculations through to the final (easy way – look at what happens to the column $\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$) and then, in order to maintain feasibility, we must have

$$\frac{4}{3} + \frac{2}{3}\Delta_1 \geq 0 \quad (1)$$

$$\frac{10}{3} - \frac{1}{3}\Delta_1 \geq 0 \quad (2)$$

$$3 - \Delta_1 \geq 0 \quad (3)$$

$$\frac{2}{3} - \frac{2}{3}\Delta_1 \geq 0 \quad (4)$$

Case 1: $\Delta_1 > 0$. **Then (1) always holds.**

(2) becomes $\frac{1}{3}\Delta_1 \leq \frac{10}{3}$; **i.e.** $\Delta_1 \leq 10$.

(3) becomes $\Delta_1 \leq 3$

(4) becomes $\frac{2}{3}\Delta_1 \leq \frac{2}{3}$; **i.e.** $\Delta_1 \leq 1$

i.e. we can have $0 < \Delta_1 \leq 1$

Case 2: $\Delta_1 < 0$ **Then (2),(3) and (4) hold no matter what value** $\Delta_1 < 0$.

(1) becomes $-\frac{4}{3} \le \frac{2}{3}\Delta_1$; i.e. $\Delta_1 \ge -2$

so we can have $-2 \le \Delta_1 < 0$

Case 3: $\Delta_1 = 0$ all inequalities hold

Combining the cases yields $$-2 \le \Delta_1 \le 1$$

B. Maximum Allowable Change in Cost Coefficients

Note: From an economic standpoint, the cost coefficients can be viewed as either marginal profit/income/cost.

1. Reddy Mikks Co. example: Suppose we replace \$3(thousand) per ton for exterior paint by \$$(3+\delta_1)$ per ton of exterior paint. What is the allowable range of δ_1 for which the current obfs remains obfs?

Tableau 0:

Basic	x_E	x_I	y_1	y_2	y_3	y_4	$-z$	Solution
$-z$	$-3-\delta_1$	-2	0	0	0	0	1	0
y_1	1	2	1	0	0	0	0	6
y_2	2	1	0	1	0	0	0	8
y_3	-1	1	0	0	1	0	0	1
y_4	0	1	0	0	0	1	0	2

Tableau 1:

Basic	x_E	x_I	y_1	y_2	y_3	y_4	$-z$	Solution
$-z$	0	$-\frac{1}{2}(1-\delta_1)$	0	$\frac{3}{2}+\delta_1$	0	0	1	$+12+4\delta_1$
y_1	0	$\frac{3}{2}$	1	$-\frac{1}{2}$	0	0	0	2
x_E	1	$\frac{1}{2}$	0	$\frac{1}{2}$	0	0	0	4
y_3	0	$\frac{3}{2}$	0	$\frac{1}{2}$	1	0	0	5
y_4	0	1	0	0	0	1	0	2

Tableau 2:

Basic	x_E	x_I	y_1	y_2	y_3	y_4	$-z$	Solution
$-z$	0	0	$\frac{1}{3}-\frac{\delta_1}{3}$	$\frac{4}{3}+\frac{2}{3}\delta_1$	0	0	1	$+12\,{}^{2}\!/_{3}+\frac{10}{3}\delta_1$
x_I	0	1	$\frac{2}{3}$	$-\frac{1}{3}$	0	0	0	$\frac{4}{3}$
x_E	1	0	$-\frac{1}{3}$	$\frac{2}{3}$	0	0	0	$\frac{10}{3}$
y_3	0	0	-1	1	1	0	0	3
y_4	0	0	$-\frac{2}{3}$	$\frac{1}{3}$	0	1	0	$\frac{2}{3}$

We could have got this last tableau with a lot less work. We could have ignored our usual rule for entering variables and just used the entering variables that we chose before changing the cost coefficient by δ_1. This is always permissible but not necessarily optimal for some choices of δ_1. Then think of not reducing the z-coeff. row until the final step. Our final tableau

would have been:

Basic	x_E	x_I	y_1	y_2	y_3	y_4	$-z$	Solution
$-z$	$-3-\delta_1$	-2	0	0	0	0	1	0
x_I	0	1	$\frac{2}{3}$	$-\frac{1}{3}$	0	0	0	$\frac{4}{3}$
x_E	1	0	$-\frac{1}{3}$	$\frac{2}{3}$	0	0	0	$\frac{10}{3}$
y_3	0	0	-1	1	1	0	0	3
y_4	0	0	$-\frac{2}{3}$	$\frac{1}{3}$	0	1	0	$\frac{2}{3}$

Now do row reduction on the z-row to bring the tableau into canonical form. It becomes:

Basic	x_E	x_I	y_1	y_2	y_3	y_4	$-z$	Solution
$-z$	0	0	$\frac{1}{3}(1-\delta_1)$	$\frac{4}{3}+\frac{2}{3}\delta_1$	0	0	1	0
x_I	0	1	$\frac{2}{3}$	$-\frac{1}{3}$	0	0	0	$\frac{4}{3}$
x_E	1	0	$-\frac{1}{3}$	$\frac{2}{3}$	0	0	0	$\frac{10}{3}$
y_3	0	0	-1	1	1	0	0	3
y_4	0	0	$-\frac{2}{3}$	$\frac{1}{3}$	0	1	0	$\frac{2}{3}$

Note that getting rid of the -2 by using the first row does not affect the coefficients of the δ_1. The only time the coefficients were affected was when we needed to get rid of $3-\delta_1$ and this used the first row below the objective function row.

If this tableau is to remain optimal, we must have all the reduced costs greater than or equal to 0; i.e.

$$\frac{1}{3} - \frac{1}{3}\delta_1 \geq 0 \quad (1) \Rightarrow \delta_1 \leq 1$$

$$\frac{4}{3} + \frac{2}{3}\delta_1 \geq 0 \quad (2) \Rightarrow \delta_1 \geq -2$$

so the allowable range of δ_1 is $-2 \leq \delta_1 \leq 1$

<u>Matrix approach:</u> Recall that $\underline{r} = \underline{c}_D - \underline{c}_B B^{-1} D$ so suppose $\underline{c}$ is replaced by $\underline{c} + \Delta\underline{c}$. We may write $\Delta\underline{c} = \Delta\underline{c}_B + \Delta\underline{c}_D$. In any event, we want

$$\underline{r} + \Delta\underline{r} = \underline{c}_D + \Delta\underline{c}_D - (\underline{c}_B + \Delta\underline{c}_B)B^{-1}D \geq \underline{0}$$

```
Max 3xE+2xI
ST

MAT_ A)   xE+2xI<6
MAT_ B)   2xE+xI<8
WKT_DEMD) -xE+xI<1
WKT_DEMD)           xI<2

END
```

```
       OBJECTIVE FUNCTION VALUE

       1)      12.66667

  VARIABLE        VALUE          REDUCED COST
        XE         3.333333          0.000000
        XI         1.333333          0.000000


       ROW   SLACK OR SURPLUS     DUAL PRICES
   MAT_ A)         0.000000          0.333333
   MAT_ B)         0.000000          1.333333
 WKT_DEMD)         3.000000          0.000000
 WKT_DEMD)         0.666667          0.000000

 NO. ITERATIONS=       1
```

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```
      ROW   SLACK OR SURPLUS     DUAL PRICES
  MAT_ A)         0.000000          0.333333
  MAT_ B)         0.000000          1.333333
WKT_DEMD)         3.000000          0.000000
WKT_DEMD)         0.666667          0.000000

NO. ITERATIONS=       1


RANGES IN WHICH THE BASIS IS UNCHANGED:

                          OBJ COEFFICIENT RANGES
VARIABLE         CURRENT        ALLOWABLE        ALLOWABLE
                   COEF          INCREASE         DECREASE
      XE        3.000000         1.000000         2.000000
      XI        2.000000         4.000000         0.500000

                          RIGHTHAND SIDE RANGES
     ROW         CURRENT        ALLOWABLE        ALLOWABLE
                   RHS           INCREASE         DECREASE
   MAT_ A       6.000000         1.000000         2.000000
   MAT_ B       8.000000         4.000000         2.000000
WKT_DEMD        1.000000         INFINITY         3.000000
WKT_DEMD        2.000000         INFINITY         0.666667


       OBJECTIVE FUNCTION VALUE

       1)      12.66667

 VARIABLE        VALUE          REDUCED COST
       XE         3.333333          0.000000
       XI         1.333333          0.000000


      ROW   SLACK OR SURPLUS     DUAL PRICES
  MAT_ A)         0.000000          0.333333
  MAT_ B)         0.000000          1.333333
WKT_DEMD)         3.000000          0.000000
WKT_DEMD)         0.666667          0.000000

NO. ITERATIONS=       1

RANGES IN WHICH THE BASIS IS UNCHANGED:

                          OBJ COEFFICIENT RANGES
VARIABLE         CURRENT        ALLOWABLE        ALLOWABLE
                   COEF          INCREASE         DECREASE
      XE        3.000000         1.000000         2.000000
      XI        2.000000         4.000000         0.500000

                          RIGHTHAND SIDE RANGES
     ROW         CURRENT        ALLOWABLE        ALLOWABLE
                   RHS           INCREASE         DECREASE
   MAT_ A       6.000000         1.000000         2.000000
   MAT_ B       8.000000         4.000000         2.000000
WKT_DEMD        1.000000         INFINITY         3.000000
WKT_DEMD        2.000000         INFINITY         0.666667
```

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10/31/2014

Example 5.1:

$$\begin{array}{lllll} \text{Max} & z = & & 2x_2 & +x_3 \\ \text{s.t.} & & & & \\ x_1 & +x_2 & +4x_3 & \leq 7 & (1) \\ -3x_1 & +x_2 & +2x_3 & \leq 3 & (2) \\ x_i & \geq 0 & i = 1,2,3 & & \end{array}$$

Your first reaction might be that x_1 does not play an active role since it is not in the objective function but that would be incorrect. Through the constraints (1) and (2), x_1 plays a role in the size of x_2 and x_3 so it is an important "player".

In standard from the LP becomes:

$$
\begin{aligned}
\textbf{Min} \quad z &= -2x_2 - x_3 \\
\text{s.t.} \quad & \\
x_1 + x_2 + 4x_3 + y_1 &= 7 \quad (1) \\
-3x_1 + x_2 + 2x_3 + y_2 &= 3 \quad (2) \\
x_i \geq 0 \quad i = 1,2,3 \qquad y_i &\geq 0 \quad i = 1,2
\end{aligned}
$$

Apply the primal simplex method:

Basic	x_1	x_2	x_3	y_1	y_2	Solution	
$-z$	0	-2	-1	0	0	0	"0"
y_1	1	1	4	1	0	7	$\frac{7}{1} = 7$
y_2	-3	1	2	0	1	3	$\frac{3}{1} = 3$
$-z$	-6	0	3	0	2	6	"1"
y_1	4	0	2	1	-1	4	
x_2	-3	1	2	0	1	3	
$-z$	0	0	6	$\frac{3}{2}$	$\frac{1}{2}$	12	
x_1	1	0	$\frac{1}{2}$	$\frac{1}{4}$	$-\frac{1}{4}$	1	
x_2	0	1	$\frac{7}{2}$	$\frac{3}{4}$	$\frac{1}{4}$	6	

Resources (1) and (2) are both scarce since $y_1 = y_2 = 0$.

Unit Worth:

optimal

Basic	x_1	x_2	x_3	y_1	y_2	
--z	0	0	6	$\frac{3}{2}$	$\frac{1}{2}$	12

Unit worth of resource (1) = $\frac{3}{2}$

Unit worth of resource (2) = $\frac{1}{2}$

Max change in resource availability:

Resource (1):

$$1 + \frac{1}{4}\Delta_1 \geq 0 \quad (1)$$

$$6 + \frac{3}{4}\Delta_1 \geq 0 \quad (2)$$

Case1: $\Delta_1 \geq 0$, (1) and (2) always true.

Case 2: $\Delta_1 < 0$ then

$$\frac{1}{4}\Delta_1 \geq -1 \quad (1) \quad \text{i.e.} \quad \Delta_1 \geq -4$$

$$\frac{3}{4}\Delta_1 \geq -6 \quad (2) \quad \text{i.e.} \quad \Delta_1 \geq -8$$

Hence

$$-4 \leq \Delta_1 < +\infty$$

and so Resource (1) ranges between +3 and $+\infty$.

Resource (2):

$$1 - \frac{1}{4}\Delta_2 \geq 0 \quad (1)$$

$$6 + \frac{1}{4}\Delta_2 \geq 0 \quad (2)$$

Case1: $\Delta_2 \geq 0$ Then (2) is always satisfied.

(1) becomes $\frac{1}{4}\Delta_2 \leq 1 \quad (1) \quad \text{i.e.} \quad \Delta_2 \leq 4$

Case2: $\Delta_2 < 0$ Then (1) is always satisfied.

(2) becomes $\frac{1}{4}\Delta_2 \geq -6$ (2) i.e. $\Delta_2 \geq -24$

Hence $-24 \leq \Delta_2 \leq 4$

so $-24 + 3 \leq \Delta_2 + 3 \leq 4 + 3$

and so Resource (1) ranges between –21 and 7.

Maximum allowable change in cost coefficients:

e.g. cost coefficient of x_2 (in max becomes $2 + \delta_2$ in std LP with min becomes $-2 - \delta_2$). We look at the min version.

x_2 row from optimal tableau

x_2	0	1	$\frac{7}{2}$	$\frac{3}{4}$	$\frac{1}{4}$	6

$$6 + \frac{7}{2}\delta_2 \geq 0 \quad (1)$$
$$\frac{3}{2} + \frac{3}{4}\delta_2 \geq 0 \quad (2)$$
$$\frac{1}{2} + \frac{1}{4}\delta_2 \geq 0 \quad (3)$$

Case 1: $\delta_2 < 0$ **Then**

$$\frac{7}{2}\delta_2 \geq -6 \Rightarrow \delta_2 \geq \frac{-12}{7} \quad (1)$$
$$\frac{3}{4}\delta_2 \geq \frac{-3}{2} \Rightarrow \delta_2 \geq -2 \quad (2) \quad \text{so } -\frac{12}{7} \leq \delta_2 < 0$$
$$\frac{1}{4}\delta_2 \geq \frac{-1}{2} \Rightarrow \delta_2 \geq -2 \quad (3)$$

Case 2: $\delta_2 = 0$ **all satisfied**

Case 3: $\delta_2 > 0$ **all satisfied**

Hence

$$-\frac{12}{7} \leq \delta_2 < +\infty$$

so the original cost coefficient of x_2 (in the max problem) can range from

$$2-\frac{12}{7}\le 2+\delta_2<2+\infty$$

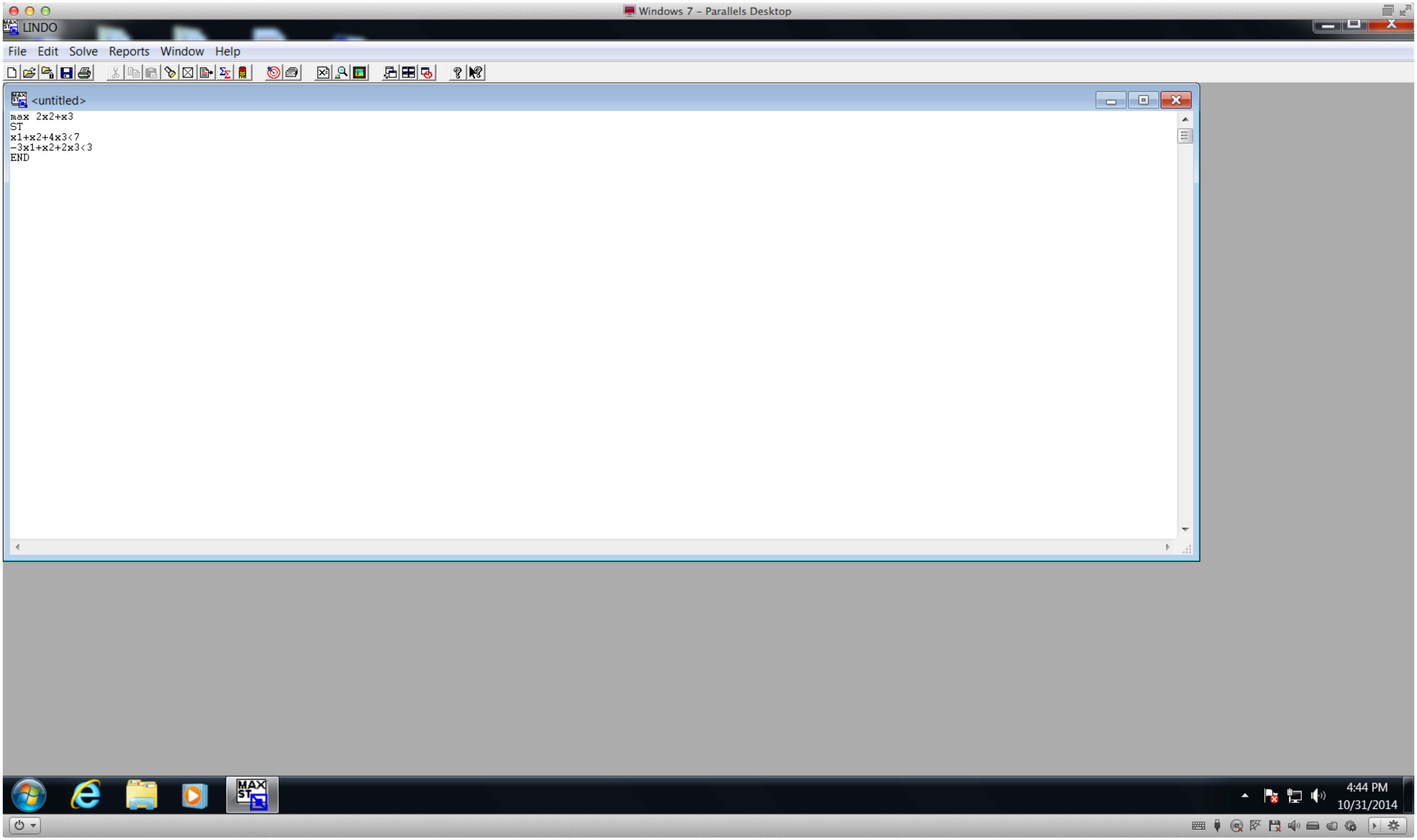
that is

$$2-\frac{12}{7}\le c_2<+\infty$$

As for the cost coefficient of x_3, we note that x_3 is not a basic variable in the final tableau so that the cost coefficient of x_3 changed by δ_3 is just $6-\delta_3$. Hence, the condition on δ_3 is just

$$6-\delta_3\ge 0\Rightarrow -\infty<\delta_3\le 6$$

Hence, for the max LP, the cost coefficient of x_3, namely c_3, can range from $-\infty+1<c_3=1+\delta_3\le 1+6$ i.e. $-\infty<c_3\le 7$.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max 2x2+x3
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x1+x2+4x3<7
-3x1+x2+2x3<3
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10/31/2014

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```
        OBJECTIVE FUNCTION VALUE

        1)      12.00000

  VARIABLE        VALUE          REDUCED COST
        X2         6.000000          0.000000
        X3         0.000000          6.000000
        X1         1.000000          0.000000


       ROW   SLACK OR SURPLUS     DUAL PRICES
        2)         0.000000          1.500000
        3)         0.000000          0.500000

 NO. ITERATIONS=       2

 LP OPTIMUM FOUND AT STEP      2

        OBJECTIVE FUNCTION VALUE

        1)      12.00000

  VARIABLE        VALUE          REDUCED COST
        X2         6.000000          0.000000
        X3         0.000000          6.000000
        X1         1.000000          0.000000


       ROW   SLACK OR SURPLUS     DUAL PRICES
        2)         0.000000          1.500000
        3)         0.000000          0.500000

 NO. ITERATIONS=       2


 RANGES IN WHICH THE BASIS IS UNCHANGED:

                           OBJ COEFFICIENT RANGES
 VARIABLE         CURRENT        ALLOWABLE        ALLOWABLE
                   COEF          INCREASE         DECREASE
       X2        2.000000        INFINITY          1.714286
       X3        1.000000        6.000000          INFINITY
       X1        0.000000        2.000000          6.000000

                           RIGHTHAND SIDE RANGES
      ROW         CURRENT        ALLOWABLE        ALLOWABLE
                    RHS          INCREASE         DECREASE
        2        7.000000        INFINITY          4.000000
        3        3.000000        4.000000         24.000000
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4:46 PM
10/31/2014

C. Matrix Form of Simplex Algorithm

1. Recall: Matrix form of LP problem

$$\begin{aligned} \min \quad z &= \underline{c} \cdot \underline{x} \\ \text{s.t.} \quad & \\ A\underline{x} &= \underline{b} \\ \underline{x} &\geq \underline{0} \end{aligned}$$

where $\underline{b}$ is an m-vector $\underline{b} \geq \underline{0}$

$\underline{c}$ is an n-vector of constants

$\underline{x}$ is an n-vector of variables

$A = (a_{ij})$ is an $m \times n$ matrix of constants r(A)=m

Write $A = \left[\begin{array}{c|c} B & D \end{array} \right]$, that is, as a partitioned matrix, where

B is an $m \times m$ matrix, r(B)=m; B is a basis for $\mathfrak{R}^m$

Write $\underline{c} = \left[\begin{array}{c|c} \underline{c}_B & \underline{c}_D \end{array} \right]$ and $\underline{x} = \left[\begin{array}{c} \underline{x}_B \\ \hline \underline{x}_D \end{array} \right]$.

Note: B need not be the *first m* columns; however, $\underline{c}_B$ corresponds to cost coefficients over the columns of B and $\underline{x}_B$ correspond to the x-variates over the columns of B.

Now $$z = \underline{c} \cdot \underline{x} = \left[\begin{array}{c|c} \underline{c}_B & \underline{c}_D \end{array}\right] \cdot \left[\begin{array}{c} \underline{x}_B \\ \hline \underline{x}_D \end{array}\right] = \underline{c}_B \cdot \underline{x}_B + \underline{c}_D \cdot \underline{x}_D$$

$$\left[\begin{array}{c|c} B & D \end{array}\right] \left[\begin{array}{c} \underline{x}_B \\ \hline \underline{x}_D \end{array}\right] = \underline{b} \text{ so } B\underline{x}_B + D\underline{x}_D = \underline{b} \Rightarrow \underline{x}_B = B^{-1}\underline{b} - B^{-1}D\underline{x}_D$$

In tableau form:

Basic	$\underline{x}_B$	$\underline{x}_D$	Solution
$-z$	$\underline{0}$	$\underline{c}_D - \underline{c}_B B^{-1} D$	$-\underline{c}_B B^{-1} \underline{b}$
$\underline{x}_B$	I_m	$B^{-1}D$	$B^{-1}\underline{b}$

2. entering and leaving conditions:

a. entering: look for most negative reduced cost in

$\underline{c}_D - \underline{c}_B B^{-1} D$; say j_0

b. leaving: $\min_{\substack{i \\ \text{s.t.} \\ (B^{-1}D)_{i j_0} > 0}} \left\{ \dfrac{(B^{-1}\underline{b})_i}{(B^{-1}D)_{i j_0}} \right\}$

3. sensitivity analysis: $B^{-1}(\underline{b} + \Delta) \geq \underline{0}$. If y_1 is a slack variable,

the column of D corresponding to it is $\begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$ so $B^{-1}\begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$ gives the coefficient of $B^{-1}\begin{pmatrix}\Delta_1\\0\\0\\\vdots\\0\end{pmatrix}$.

Example 5.2:

$$\begin{array}{llll} \min & z = & -5x_1 - 12x_2 - 4x_3 & \\ \text{st} & & & \\ x_1 + 2x_2 + x_3 & \le & 10 & (1) \\ 2x_1 - x_2 + 3x_3 & = & 8 & (2) \\ x_i \ge 0 & i = 1,2,3 & & \end{array}$$

Basic	x_1	x_2	x_3	y_1	r_2	Solution	Phase 1
$-z$	0	0	0	0	1	0	
y_1	1	2	1	1	0	10	
r_2	2	-1	3	0	1	8	
$-z$	-2	1	-3	0	0	-8	"0"
y_1	1	2	1	1	0	10	$\frac{10}{1} = 10$
r_2	2	-1	3	0	1	8	$\frac{8}{3}$
$-z$	-5	-12	-4	0	0	0	end of Phase 1
y_1	$\frac{1}{3}$	$\frac{7}{3}$	0	1	$-\frac{1}{3}$	$\frac{22}{3}$	
x_3	$\frac{2}{3}$	$-\frac{1}{3}$	1	0	$\frac{1}{3}$	$\frac{8}{3}$	

$-z$	$-\frac{7}{3}$	$-\frac{40}{3}$	0	0	$\frac{4}{3}$	$\frac{32}{3}$	**Phase 2**
y_1	$\frac{1}{3}$	$\frac{7}{3}$	0	1	$-\frac{1}{3}$	$\frac{22}{3}$	"0"
x_3	$\frac{2}{3}$	$-\frac{1}{3}$	1	0	$\frac{1}{3}$	$\frac{8}{3}$	
$-z$	$-\frac{3}{7}$	0	0	$\frac{40}{7}$	$-\frac{4}{7}$	$\frac{368}{7}$	"1"
x_2	$\frac{1}{7}$	1	0	$\frac{3}{7}$	$-\frac{1}{7}$	$\frac{22}{7}$	$\frac{22/7}{1/7} = 22$
x_3	$\frac{5}{7}$	0	1	$\frac{1}{7}$	$\frac{2}{7}$	$\frac{26}{7}$	$\frac{26/7}{5/7} = \frac{26}{5}$
$-z$	0	0	$\frac{3}{5}$	$\frac{29}{5}$	$-\frac{2}{5}$	$\frac{274}{5}$	"2"
x_2	0	1	$-\frac{1}{5}$	$\frac{2}{5}$	$-\frac{1}{5}$	$\frac{12}{5}$	optimal
x_1	1	0	$\frac{7}{5}$	$\frac{1}{5}$	$\frac{2}{5}$	$\frac{26}{5}$	

B at optimal tableau (basis is x_2, x_1 in that order) is

$$\mathbf{B} = \begin{pmatrix} 2 & 1 \\ -1 & 2 \end{pmatrix} \text{ while from the final tableau } \mathbf{B}^{-1} = \begin{pmatrix} \frac{2}{5} & -\frac{1}{5} \\ \frac{1}{5} & \frac{2}{5} \end{pmatrix}$$

END OF SECTION 5